

# Calculations of the dynamical critical exponent using the asymptotic series summation method

V.V. Prudnikov\*, P.V. Prudnikov and A.S. Krinitsyn

*Dept. of Theoretical Physics, Omsk State University 55a, Pr. Mira, 644077, Omsk, Russia*

We consider how the Padé–Borel, Padé–Borel–Leroy, and conformal mapping summation methods for asymptotic series can be used to calculate the dynamical critical exponent for homogeneous and disordered Ising-like systems.

## I. INTRODUCTION

This paper shows in detail how the summation methods for asymptotic series can be used to calculate the critical exponent  $z$  determining the critical slowing down of the system relaxation time  $\tau \sim \xi^z \sim |T - T_c|^{-z\nu}$  near the temperature  $T_c$  of the second-order phase transition ( $\xi$  is the correlation length and  $\nu$  is the correlation length exponent). The most complicated and interesting direction in the theory of critical phenomena is the study of dynamical processes near a critical point. A significant achievement of the renormalization group (RG) approach in studying the static critical phenomena is the constructed concept of universality classes of the critical behavior of different systems characterized by similar critical properties. The universality of the dynamical critical phenomena in contrast to the static phenomena is considerably weaker, which is manifested in the existence of a wide range of models of the critical dynamics with different dynamical critical behavior [1] and common critical properties in equilibrium. This difficulty is partially compensated because the dynamical critical characteristics of many of these models can be expressed in terms of the characteristics of their static critical behavior. This primarily concerns the models where the conservation laws hold for the order parameter or any other long-lived excitations, in particular, the energy density [1]. Because of the presence of preserved quantities, the purely hydrodynamic terms determining (and sometimes suppressing) the influence of the fluctuation processes on the dynamical (relaxation) behavior can appear in the stochastic equations of motion. The critical exponents describing the dynamical critical behavior can

---

\* E-mail : prudnikov@univer.omsk.su

be calculated in this case based on the static exponents with the same accuracy as the latter [1].

But there exist dynamical models such that the description of their critical behavior is not a trivial problem. They are primarily models with purely dissipative equations of motion. To study the critical behavior of such models, special methods must be developed, and dynamical quantities must be calculated independently of the static quantities. Among such models, the Ginzburg–Landau dynamical model (model *A* in the Halperin–Hohenberg classification scheme [1]) is most interesting; this model was first proposed by Landau and Khalatnikov to describe the anomalous sound attenuation in helium near a  $\lambda$ -point. The dynamical critical behavior of other real systems such as the Ising-like magnets later came to be described in the framework of this model.

The critical dynamics of model *A* was studied in [2] using the RG methods based on the  $\varepsilon$ -expansion. To describe this dynamics, one of us first developed a field theory approach directly for the three- and twodimensional homogeneous systems successively in the three-loop and four-loop approximations [3], [4] and then also for structurally disordered three-dimensional Ising-like systems in the three-loop approximation [5]. But the dynamical critical characteristics were determined in these papers with an accuracy considerably less than the accuracy of the description of the static critical behavior of these systems in the six-loop approximation of the theory in [6, 7, 8, 9]. This is primarily due to a fast increase in the volume of calculations already for the lowest-order perturbation series, especially in the description of the critical dynamics of structurally disordered systems. In this connection, the requirements for the accuracy of the summation methods for the asymptotic series in this theory used to calculate the value of the dynamical critical exponent  $z$  in the description of the critical dynamics of model *A* in homogeneous and disordered systems become more rigid. Here, we first present the results obtained by summing the series in this theory using the Padé-Borel- Leroy (PBL) and conformal mapping (CM) summation methods to calculate the values of the exponent  $z$ . We compare the obtained results with experimental data and the results of a Monte Carlo computer simulation of the critical dynamics.

## II. MODEL

The field theory RG method, which permits calculating the values of the critical exponents characterizing the asymptotic behavior of the thermodynamic and correlation functions near the critical temperature, is widely used to describe the anomalous properties of the thermodynamic characteristics of systems in the second-order phase transitions. Our model of the critical behavior of a homogeneous ferromagnetic system is the classical spin system thermodynamically equivalent to the  $O(n)$ -symmetric Ginzburg–Landau model with the effective Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int d^d x \left( |\nabla \varphi|^2 + r_0 \varphi^2 + \frac{g_0}{12} \varphi^4 \right), \quad (1)$$

where  $d$  is the dimension of the system,  $\varphi(\vec{x}, t)$  is the  $n$ -component order parameter (magnetization),  $r_0 \sim T - T_{c0}$  ( $T_{c0}$  is the critical temperature in the mean-field approximation), and  $g_0 > 0$  is the vertex of the magnetization fluctuation interaction. The dynamical behavior of a magnet in the relaxation regime near a critical point in the framework of model  $A$  is described by the Langevin-type kinetic equation for the order parameter,

$$\frac{\partial \varphi}{\partial t} = -\lambda_0 \frac{\delta \mathcal{H}}{\delta \varphi} + \zeta + \lambda_0 h, \quad (2)$$

where  $\lambda_0$  is the kinetic coefficient,  $\zeta(\vec{x}, t)$  is a Gaussian random force, and  $h(\vec{x}, t)$  is an external magnetic field. It is well known that its solution in the form of correlation and response functions can be obtained using a generating functional of the form

$$\Omega = \int D[\varphi] D[\tilde{\varphi}] \exp \left( -\mathcal{H}_{eff}[\varphi, \tilde{\varphi}] + \int (\varphi h + \tilde{\varphi} \tilde{h}) d^d x dt \right), \quad (3)$$

where an auxiliary field  $\tilde{\varphi}$  with the field source  $\tilde{h}$  and the action functional

$$\mathcal{H}_{eff} = \int d^d x dt \left( \lambda_0^{-1} \tilde{\varphi}^2 + i \tilde{\varphi} \left( \lambda_0^{-1} \frac{\partial \varphi}{\partial t} + \frac{\delta \mathcal{H}}{\delta \varphi} \right) \right). \quad (4)$$

are introduced. The functions of the order parameter response to the field  $h$  are in this case determined as

$$G(x, t) = \frac{\delta \langle \varphi(x, t) \rangle}{\delta h(x, t)} \Big|_{h=0} = \frac{1}{\Omega} \frac{\delta^2 \Omega}{\delta h(x, t) \delta \tilde{h}(0, 0)} = \langle \varphi(x, t) \tilde{\varphi}(0, 0) \rangle. \quad (5)$$

Instead of the response function, it is more convenient to consider its vertex part  $\Gamma^{(1,1)}(k, \omega)$ .

The long-range long-lived fluctuations in the order parameter, which mainly determine the

anomalies of the equilibrium and nonequilibrium characteristics of systems as  $T \rightarrow T_c$ , are taken into account in the framework of the RG methods developed in [1, 10]. The  $(N + \tilde{N})$ th-order renormalized dynamical vertex functions  $\Gamma_R^{(N, \tilde{N})}(k, \omega)$ , which are used in the theory and uniquely determine all the observable characteristics of the system, are given by the RG differential equation

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - r \gamma_r \frac{\partial}{\partial r} - \lambda \gamma_\lambda \frac{\partial}{\partial \lambda} - \frac{(N + \tilde{N})}{2} \gamma_\varphi \right] \Gamma_R^{(N, \tilde{N})} = 0 \quad (6)$$

with the renormalized charge  $g$ , the reduced temperature  $r$ , and the kinetic coefficient  $\lambda$ . The asymptotic behavior of the thermodynamic and correlation functions, which have a power-law character as  $T \rightarrow T_c$ , is determined by the zero  $g^*$  of the function  $\beta(g)$ :  $\beta(g^*) = 0$ , and by the function values  $\gamma_r(g^*)$ ,  $\gamma_\lambda(g^*)$  and  $\gamma_\phi(g^*)$  specifying the static and dynamical critical exponents:  $\nu = (2 + \gamma_r(g^*))^{-1}$ ,  $\eta = \gamma_\varphi(g^*)$ ,  $z = 2 + \gamma_\lambda(g^*)$  (for example, if  $\gamma = \nu(2 - \eta)$ , then the magnetic susceptibility behaves as  $\chi \sim |T - T_c|^{-\gamma}$ ).

The functions  $\beta(g)$ ,  $\gamma_r(g)$ ,  $\gamma_\lambda(g)$  and  $\gamma_\phi(g)$  contained in the differential RG equation can be calculated as series in  $g$ . If the space dimension  $d$  is close to four, then the coordinate of the fixed point  $g^*$  of the function  $\beta(g)$  takes small values. The methods of the perturbation theory in the coupling constant  $g \sim 4 - d$  can be used in this case, and the critical exponents can be calculated. For real systems with  $d = 3, 2$  the series in  $g$  are asymptotic series, and they can be summed using special methods not based on the perturbation theory representations [6, 7].

In the description of the critical behavior of structurally disordered systems with frozen nonmagnetic admixture atoms or vacancies at the lattice nodes, which play the role of point defects of the structure, the additional term

$$\Delta \mathcal{H}[\varphi, V] = \frac{1}{2} \int d^d x V(x) \varphi^2, \quad (7)$$

where  $V(x)$  is the potential of the random field of defects, is introduced in the effective Hamiltonian of Ginzburg–Landau model (1), which leads to fluctuations in the local critical temperature  $r_0 \sim T - T_{c0}$ . The distribution of the structure defects over the volume of the system is assumed to be Gaussian (taking the deviations from the Gaussian distribution into account gives only corrections inessential in the critical domain) with the distribution function

$$P[V] = A_V \exp \left[ -(16v_0)^{-1} \int d^d x V^2(x) \right], \quad (8)$$

where  $A_V$  is the normalization factor and  $v_0$  is a positive parameter proportional to the concentration of defects and to their squared potential. The correlation functions and the response functions for structurally disordered systems, obtained later using the generating functional, must be averaged over the random defect potential  $V(x)$ . This averaging can be performed most effectively using the replica method, which is widely used for similar purposes in the study of weakly disordered systems (see, e.g., [11]). The essence of this method is that the following mathematical transformation is formally used to obtain the free energy of a disordered system in averaging over the defect distribution:

$$-F/T = \langle \ln Z \rangle = \lim_{m \rightarrow 0} \frac{1}{m} \ln \langle Z^m \rangle = \lim_{m \rightarrow 0} \frac{1}{m} \ln \text{Sp}_{\{\varphi_i\}} \exp(-\hat{H}_{\text{repl}}[\{\varphi_i\}]/T), \quad (9)$$

where the double angle brackets denote the average over the probability distribution  $P[V]$  of different defect configurations,  $Z$  is the partition function of the original disordered system, and  $m$  replicas ("images") of the original field  $\varphi$  -  $\{\varphi_i\}$  are introduced with the replica indices  $i = 1, \dots, m$ . The replica Hamiltonian is then determined as

$$\exp(-\hat{H}_{\text{repl}}^{(m)}[\{\varphi_i\}]/T) = \left\langle \left\langle \prod_{i=1}^m \exp(-\hat{H}[\{\varphi_i\}]/T) \right\rangle \right\rangle, \quad (10)$$

and is translation invariant, in contrast to the original Hamiltonian. Applying this averaging procedure to dynamical generating functional (3), we can obtain the replica action functional

$$\begin{aligned} \hat{H}_{\text{repl}}^{(m)} = & \sum_i \int d^d x dt \left[ -\lambda_0^{-1} \tilde{\varphi}_i \tilde{\varphi}_i + i \tilde{\varphi}_i \left( \lambda_0^{-1} \frac{\partial \varphi_i}{\partial t} - \nabla^2 \varphi_i + r_0 \varphi_i \right) + \frac{i}{3!} g_0 \tilde{\varphi}_i \varphi_i \varphi_i \varphi_i \right] + \\ & + 4v_0 \sum_{i,j} \int d^d x dt dt' \tilde{\varphi}_i(x, t) \varphi_i(x, t) \tilde{\varphi}_j(x, t') \varphi_j(x, t'). \end{aligned} \quad (11)$$

instead of the action functional for homogeneous model (4). According to the replica method, the properties of the original disordered system are obtained in the limit as  $m \rightarrow 0$ . This limit annihilates the connected diagrams containing loops formed by the admixture vertex  $v_0$ .

Taking the structural disorder of systems into account in describing the critical behavior thus leads to introducing an additional interaction vertex  $v_0$  in the effective Hamiltonian. This vertex determines the effects of interaction of fluctuations of the  $nm$ -component order parameter in terms of the field of defects. The subsequent procedure for renormalizing the dynamical vertex functions is given by the RG differential equation

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \beta_v \frac{\partial}{\partial v} - r \gamma_r \frac{\partial}{\partial r} - \lambda \gamma_\lambda \frac{\partial}{\partial \lambda} - \frac{(N + \tilde{N})}{2} \gamma_\varphi \right] \Gamma_R^{(N, \tilde{N})} = 0 \quad (12)$$

with the renormalization charges  $g$  and  $v$ . The functions  $\beta_g$  and  $\beta_v$  in this case depend on  $g$  and  $v$  as on model parameters. The fixed points  $(g^*, v^*)$  of the RG transformations are determined by the zeros of the functions  $\beta_g$  and  $\beta_v$ ,  $\beta_g(g^*, v^*) = 0$ ,  $\beta_v(g^*, v^*) = 0$ , and the critical exponents are determined by the values of the functions  $\gamma_r(g^*, v^*)$ ,  $\gamma_\varphi(g^*, v^*)$  and  $\gamma_\lambda(g^*, v^*)$  at the corresponding stable fixed points of the RG transformation.

As in the case of homogeneous systems, the functions  $\beta_g$ ,  $\beta_v$ ,  $\gamma_r$ ,  $\gamma_\lambda(g)$  and  $\gamma_\phi(g)$  can be calculated as series in  $g$  and  $v$  only if the system dimension  $d$  is close to four. For real systems with  $d = 3$ , the series in  $g$  and  $v$  are only asymptotically convergent.

### III. SUMMATION METHODS FOR ASYMPTOTIC SERIES

To discuss the dynamical critical behavior of homogeneous systems and to calculate the exponent  $z$ , we only need the functions  $\beta(g)$  and  $\gamma_\lambda(g)$ . The explicit form of the first of them in the six-loop approximation for three-dimensional systems and in the four-loop approximation for two-dimensional systems was obtained in [6, 7]. We calculated the dynamical scaling functions  $\gamma_\lambda$  in the four-loop approximation for two- and three-dimensional Ising models [4]. As a result, the functions  $\beta(g)$  and  $\gamma_\lambda(g)$  can be represented as the series in the charge  $g$

$$\begin{aligned} \frac{\beta(g)}{g} &= -1 + g - 0.716174g^2 + 0.930767g^3 - 1.582388g^4, \\ \gamma_\lambda(g) &= 0.027053g^2 - 0.004184g^3 + 0.022130g^4, \end{aligned} \quad (13)$$

for the two-dimensional model and

$$\begin{aligned} \frac{\beta(g)}{g} &= -1 + g - 0.422497g^2 + 0.351070g^3 - 0.376527g^4 + 0.495548g^5 - 0.749689g^6, \\ \gamma_\lambda(g) &= 0.008399g^2 - 0.000045g^3 + 0.020423g^4, \end{aligned} \quad (14)$$

for the three-dimensional model. Based on the method proposed by Lipatov [12], it was shown in [13] that although the general term of the series for the function  $\beta(g)$  increases factorially,

$$\beta(g) = \sum_{n=0}^{\infty} c_n g^n, \quad c_n \approx c(-a)^n n^b n! [1 + O(1/n)], \quad (15)$$

the series nevertheless satisfies the asymptotic convergence condition

$$\left| \beta(g) - \sum_{n=0}^N c_n g^n \right| \leq C \sigma^{N+1} [(N+1)!]^\sigma |g|^{N+1} \quad (16)$$

where  $C$  and  $\sigma$  are some constants, in the wedge-shaped domain  $|\arg g| \leq \frac{\pi}{2}\sigma$  on the complex plane. Matching asymptotic expansion (15) and the values of the first-order coefficients gives information about all the terms of the series and permits reconstructing the function  $\beta(g)$  approximately, but this requires a special summation procedure for asymptotically convergent series. Special summation methods for such series were developed in [6, 7, 8, 9, 14, 15, 16]; the most effective of them are the Padé-Borel (PB), PBL, and CM methods.

In [17], the Lipatov technique for estimating higher-order terms in this theory was generalized to the critical dynamics, and it was proved that the Lipatov asymptotic expansion also holds for the dynamical RG function  $\gamma_\lambda$ . Therefore, the above summation methods, which were successfully used to analyze the static RG functions, can also be used to analyze the dynamical RG functions.

For the three-dimensional disordered Ising model, the obtained representations of the functions  $\beta_g$ ,  $\beta_v$  and  $\gamma_\lambda$  as series in the charges  $g$  and  $v$  have the forms

$$\begin{aligned}\frac{\beta_g}{g} &= -1 + g + \frac{3}{2}v - \frac{308}{729}g^2 - \frac{104}{81}gv - \frac{185}{216}v^2 + \sum_{i+j \geq 3} b_{i,j}^{(g)} g^i v^j, \\ \frac{\beta_v}{v} &= -1 + \frac{2}{3}g + v - \frac{92}{729}g^2 - \frac{50}{81}gv - \frac{95}{216}v^2 + \sum_{i+j \geq 3} b_{i,j}^{(v)} g^i v^j, \\ \gamma_\lambda &= \sum_{i,j=0}^3 \gamma_{i,j} g^i v^j = -\frac{1}{4}v + 0.008400g^2 + 0.030862gv + 0.053240v^2 - \\ &\quad - 0.012642g^3 - 0.041167g^2v - 0.152964gv^2 - 0.049995v^3,\end{aligned}\tag{17}$$

where the coefficients  $b_{i,j}^{(g)}$ ,  $b_{i,j}^{(v)}$  and  $\gamma_{i,j}$  were calculated in [9] and [5] in the respective three-loop and six-loop approximations.

As a preliminary illustration of the PB, PBL, and CM summation methods, we consider the exactly solvable problem of determining the ground state energy of the linear anharmonic oscillator with the Hamiltonian

$$\mathcal{H} = p^2 + x^2 + gx^4.\tag{18}$$

This permits detecting special features of the use of each of the summation methods and understanding how the accuracy of the used approximation depends on the number of terms present in the series. According to [18], the oscillator ground state energy  $E_0(g)$  can be

represented as a perturbation series in the anharmonicity constant  $g$  whose first few terms have the form

$$E_0(g) = \sum_{n=0}^{\infty} c_n g^n = 1 + \frac{3}{4}g - \frac{21}{16}g^2 + \frac{333}{64}g^3 - \frac{30885}{1024}g^4 + \frac{916731}{4096}g^5 - \frac{65518401}{32768}g^6 + \dots, \quad (19)$$

and exact values of  $E_0$  depending on  $g$  are given in [19]. It was shown in [18] that the general term of series (19) increases factorially as in expression (15).

The PB method is based on the idea that a series of form (19) can be written as the integral

$$f(g) = \int_0^{\infty} e^{-t} B(gt) dt, \quad B(g) = \sum_{n=0}^{\infty} B_n g^n, \quad B_n = \frac{c_n}{n!}. \quad (20)$$

In this expression,  $B(g)$  is called the Borel image. According to asymptotic expansion (15), the Borel image converges in a disk of radius  $1/a$ . We then apply the Padé approximation to  $B(g)$ , which consists in using a rational function of the form

$$[L/M] = \frac{\sum_{i=0}^L a_i g^i}{\sum_{j=0}^M b_j g^j} \quad (M \geq 1), \quad (21)$$

whose expansion in a Taylor series (in a neighborhood of the point  $g = 0$ ) coincides with the expansion of the Borel image as closely as possible. A function of form (21) has  $L + 1$  coefficients in the numerator and  $M + 1$  coefficients in the denominator. The entire set of coefficients is thus determined up to the common factor; to be definite, we usually set  $b_0 = 1$ . As a result, we obtain the total set of  $L + M + 1$  free parameters. This means that the coefficients of the expansion of the function  $[L/M]$  in a Taylor series must generally coincide with the corresponding coefficients of series (19). If the number of terms  $N$  in the series is finite, then the approximants  $[L/M]$  must satisfy the condition  $L + M \leq N$ .

A generalization of the PB method is the PBL method for which formulas (20) can be written in the extended form as

$$f(g) = \int_0^{\infty} t^b e^{-t} B(gt) dt, \quad B(g) = \sum_{n=0}^{\infty} B_n g^n, \quad B_n = \frac{c_n}{\Gamma(n + b + 1)}, \quad (22)$$

where  $b$  is an arbitrary parameter. In the simplest realization of the method, we consider the values  $b = 0$  and  $b = 1$ , for example, and then analyze the variations in the series approximations resulting from variations in the parameter  $b$ , which permits estimating the accuracy of approximations obtained by the PB and PBL methods. In our case where the



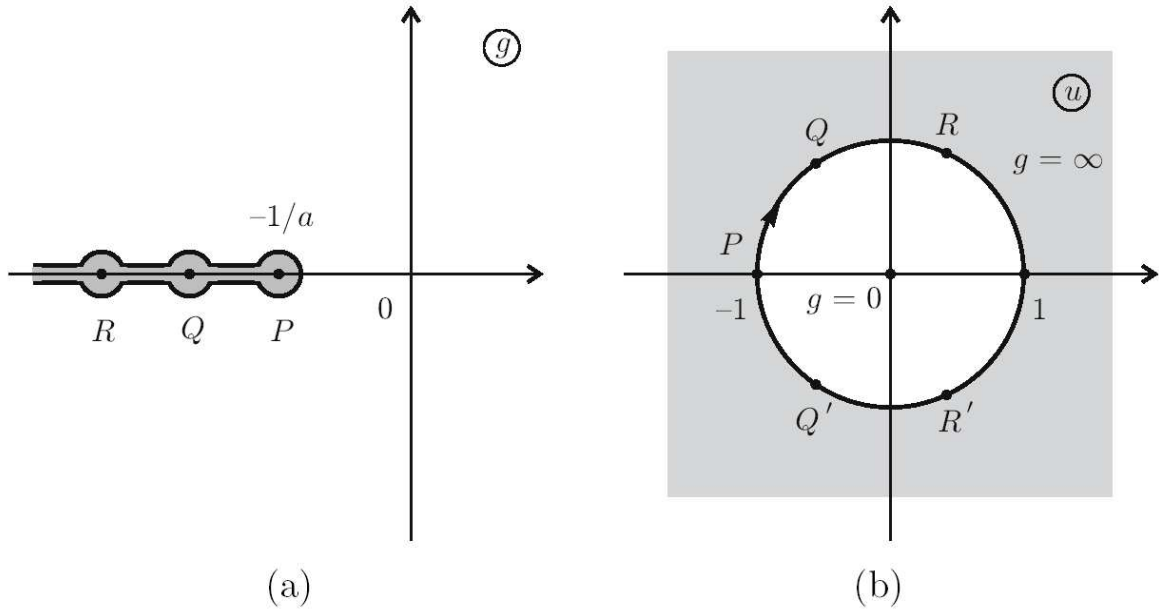


FIG. 1: Domain of the Borel image analyticity on the complex plane with the cut  $(-\infty, -1/a)$  along the real axis **(a)** and its conformal mapping to the unit disk **(b)**.

PBL method is applied to the series determining the characteristics of the critical behavior of homogeneous and disordered systems, for a test series, we choose series (19) corresponding to the exactly solvable problem of calculating the anharmonic oscillator energy with the asymptotic convergence of the series, which is similar to that of series (13) and (14) in the theory of critical phenomena. Therefore, applying the PBL method to series (13) and (14) in the theory of critical phenomena, we plan to use the values of the parameter  $b$  that lead to the best approximations of series (19).

We consider the main principles of applying the CM method to the Borel image (for  $b = 0$ ). According to asymptotic expansion (15), the Borel image is an analytic function in the complex plane of  $g$  with the cut from  $-1/a$  to  $-\infty$  (Fig. 1a). Because the integration in formulas (22) is performed along the entire real axis, we must analytically continue  $B(g)$  to arbitrary complex  $g$  beyond the convergence disk  $|g| < 1/a$ , and this continuation is realized by the conformal map  $g = w(u)$  with

$$w(u) = \frac{4}{a} \frac{u}{(1-u)^2}, \quad u = \frac{(1+ag)^{1/2} - 1}{(1+ag)^{1/2} + 1}, \quad (23)$$

which takes the plane with a cut to the unit disk  $|u| < 1$  (Fig. 1b). The representation of

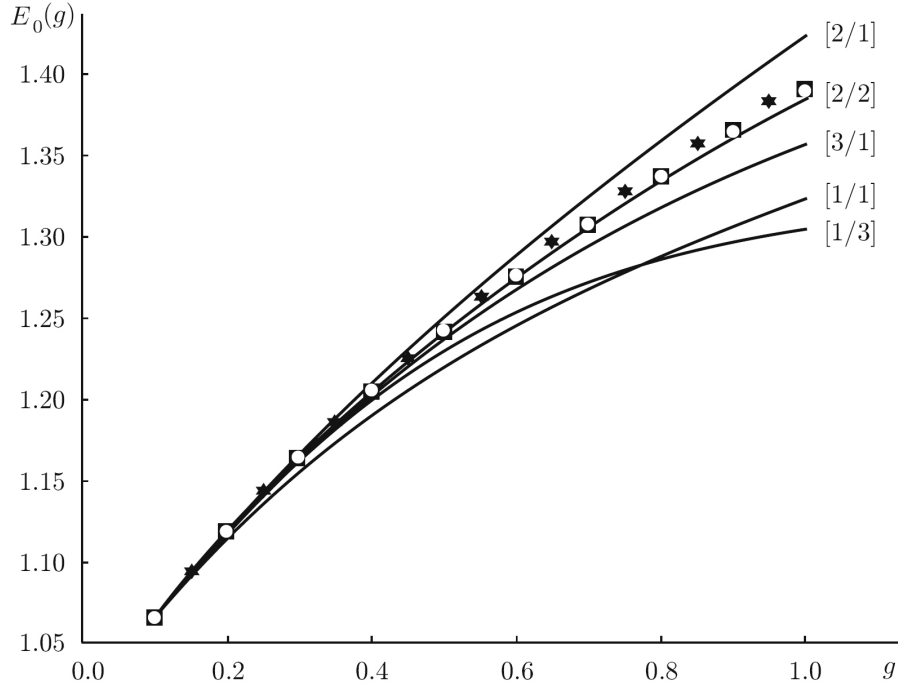


FIG. 2: Comparison of the exact values of the anharmonic oscillator ground state energy  $E_0$  (black squares) for different values of the anharmonicity constant  $g$  with the results obtained by the PB summation method for different  $[L/M]$  (solid lines), the PBL method with the parameter  $b = 2.221426$  for the approximant  $[1/1]$  (white circles), and the CPB method with the constant  $a = 3$  (stars).

$B(g)$  as a series in  $u$  thus leads to the convergent series for any  $g$ :

$$B(g) = \sum_{n=0}^{\infty} B_n g^n \Big|_{g=w(u)} \quad \Longrightarrow \quad B(u) = \sum_{n=0}^{\infty} U_n u^n. \quad (24)$$

Indeed, all possible singular points  $P, Q, R, \dots$  of the function  $B(g)$  lie along the cut, and their images  $P, Q, Q', R, R', \dots$  lie on the boundary of the disk  $|u| = 1$ . Hence, the second of series (24) converges for all  $|u| < 1$ . The relationship between the coefficients  $U_n$  and  $B_n$  can be expressed as

$$U_0 = B_0, \quad U_n = \sum_{m=1}^n B_m \left( \frac{4}{a} \right)^m C_{n+m-1}^{n-m} \quad (n \geq 1). \quad (25)$$

These formulas solve the above problem as follows: the Borel image  $B(u(g))$  converges for any singular points  $g = \infty, g = -1/a$  and  $g = g_0$  with  $g_0 \in (-\infty, -1/a)$ , and its coefficients  $U_n$  are related to the original coefficients  $c_n$  and the parameter  $b$  by

TABLE I: Sums of squared deviations of the values of the oscillator energy  $E_0$  calculated using the PB, PBL, and CPB methods from the exact values of  $E_0$  (the gaps show that the Borel image has a pole when the corresponding approximant is used).

[L/M]	[1/1]	[1/2]	[2/1]	[1/3]	[2/2]	[3/1]	[1/4]	[2/3]
$S_{\text{PB}}^2$	0.013923	-	0.002703	0.016687	0.000105	0.002535	-	-
$b$	2.221426	1.582184	1.466092	1.276639	1.131538	1.194466	2.016677	1.050884
		3.441754		3.879020			4.083434	
$S_{\text{PBL}}^2$	0.000011	0.000006	0.000003	0.000007	$3 \cdot 10^{-7}$	0.000001	0.000010	0.000044
		0.000010		0.000008			0.000007	
$S_{\text{CPB}}^2$	0.000175	0.041795	0.000836	0.000307	0.000192	-	-	0.000011

[L/M]	[3/2]	[4/1]	[0/6]	[1/5]	[2/4]	[3/3]	[4/2]	[5/1]
$S_{\text{PB}}^2$	0.000001	0.004142	2.609852	0.031233	-	-	0.000247	0.000833
$b$	1.005754	1.068052	$\nexists$	2.445345	0.970348	0.959599	0.964316	1.006032
				4.194370		2.003840		
$S_{\text{PBL}}^2$	$3 \cdot 10^{-7}$	0.000004		0.000007	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$	0.000008
				0.000006		$2 \cdot 10^{-7}$		
$S_{\text{CPB}}^2$	-	-	0.000064	-	$3 \cdot 10^{-7}$	$5 \cdot 10^{-7}$	0.003932	0.000003

linear transformation (25) (see formulas (22)). Because we have  $|u| < 1$  for any  $g$ , the Padé approximation can be effectively applied to the series in the variable  $u$ . This procedure is called the conformal Padé-Borel (CPB) method.

In Fig. 2, we compare the approximations applied to the series for the anharmonic oscillator ground state energy with the exact values of  $E_0$  obtained using the PB method with different types of the Padé approximant  $[L/M]$  and the PBL and CPB methods (for  $a = 3$  [18]) with the approximant  $[1/1]$ . Fig. 2 shows that the approximation error of the PB method increases strongly as  $g$  increases, although the situation becomes better as the number  $N$  of the series terms taken into account increases and the diagonal and nearly diagonal approximants are used. But the PBL method for  $b = 2.221426$  (in expression (26)) and the CPB method even for the approximant  $[1/1]$  give results close to the exact values of  $E_0$  in the considered interval of the variable  $g$ . These results can be compared in accuracy only

TABLE II: Values of the fixed point  $g^*$  and the dynamical critical exponent  $z$  for the two- and three-dimensional Ising models obtained by the PB, PBL, and CPB methods.

		[L/M]	[1/1]	[1/2]	[2/1]	[1/3]	[3/1]	[1/4]	[2/3]
d=2	PB	$g^*$	2.1971	-	1.8210	1.8271	1.7702		
		$z$	2.1262		2.0868	2.0873	2.0820		
	PBL	$g^*$	1.6978	1.7098	-	1.7174	1.6849		
		$z$	2.0753	2.0764		2.0771	2.0742		
	CPB	$g^*$	2.1448	-	1.9137	-	1.8193		
		$z$	2.1270		2.1018		2.0922		
d=3	PB	$g^*$	1.5508	-	1.4314	1.4230	1.4175	1.4427	1.4753
		$z$	2.0202		2.0172	2.0170	2.0169	2.0175	2.0183
	PBL	$g^*$	1.3972	1.4019	1.4671	1.4056	1.4049	1.4073	-
		$z$	2.0166	2.0165		2.0166	2.0166	2.0166	
	CPB	$g^*$	1.5385	-	1.4552	-	1.4267	1.4253	-
		$z$	2.0452		2.0393		2.0374	2.0373	
		[L/M]	[3/2]	[4/1]	[1/5]	[2/4]	[3/3]	[4/2]	[5/1]
d=3	PB	$g^*$	1.4234	1.4292	-	1.4263	-	1.4200	1.3854
		$z$	2.0170	2.0172		2.0171		2.0169	2.0161
	PBL	$g^*$	1.4234	1.4355	1.4082	-	1.4221	1.4208	1.3848
		$z$	2.0170	2.0173	2.0167		2.0170	2.0170	2.0161
	CPB	$g^*$	1.4242	1.4245	-	-	-	-	1.4147
		$z$	2.0372	2.0373					2.0366

with the results obtained using the "best" diagonal approximant [2/2] in the PB method.

In realizing the PBL method, it is more convenient to perform the integral Borel transformation as

$$f(g) = \int_0^\infty dt e^{-t} B(gt^b), \quad B(g) = \sum_{n=0}^\infty B_n g^n, \quad B_n = \frac{c_n}{\Gamma(bn + 1)}, \quad (26)$$

where the parameter  $b$  for all  $[L/M]$  is determined by the requirement that the mean squared deviation of approximations from the exact values of  $E_0$  be minimum on the entire interval

of  $g$ . Using expression (26) allows approximating the exact value of  $E_0$  much better than using (22). Moreover, this transformation, in contrast to transformation (22), allows avoiding the appearance of singularities in the integrand when the abovementioned variational method for choosing the values of  $b$  is used.

Table I presents values of the sum of squared deviations  $S^2$  for the values of the oscillator energy  $E_0$  when different approximations of the exact values of  $E_0$  are used in the PB, PBL, and CPB methods. In Table I, we do not present the results obtained using approximants of the form  $[0/N]$ , because it was found that for even  $N$ , using them leads to very large deviations of the series sum from the exact solution and the deviation  $S^2$  does not have a minimum in  $b$  (only a small decrease in  $S^2$  is observed as  $b$  increases), while for odd  $N$ , singularities unremovable for any value of  $b$  appear in the Borel image corresponding to this approximant. For each of the approximants of the form  $[1/N - 1]$ , we found two values of the parameter  $b$  characterized by close values of the deviation  $S^2$ . Comparing the given values of  $S^2$  clearly shows that the accuracy of the PBL method is much higher than that of the PB method. The PB method permits obtaining results comparable in accuracy to the results obtained by the PBL method only starting from the fifth-order nearly diagonal approximants. This makes the PBL method preferable for analyzing the short series obtained in the description of the critical dynamics of homogeneous systems (13), (14) and disordered systems (17). We also note that the CPB method is second in accuracy among the methods considered and is as good in accuracy as the PBL method starting from the fifth-order approximants in  $N$  (except for the case of the approximant  $[4/2]$ ). The PB method, although it is below the CPB method in accuracy for short series, permits obtaining comparable results only for the diagonal approximant  $[2/2]$  and the nearly diagonal approximants  $[3/2]$  and  $[4/2]$ .

We now consider how the above methods can be used to calculate the dynamical critical exponent  $z$  for the homogeneous Ising model describing the critical behavior of systems with the dimensions  $d = 2$  and  $d = 3$ . As previously noted, the exponent  $z$  is determined when the functions  $\beta(g)$  and  $\gamma_\lambda(g)$  are used. The fixed point  $g^*$  of the RG transformations is found from the condition  $\beta(g^*) = 0$ . In Table II, we present the values of the fixed point  $g^*$  and the dynamical critical exponent  $z$  for the two- and three-dimensional Ising models, which were obtained by the PB, PBL (for the values of the parameter  $b$  related to the corresponding approximants in the oscillator problem), and CPB methods, where, according to [6, 7], the

constant  $a$  for the last method was taken to be 0.238659217 for  $d = 2$  and 0.14777422 for  $d = 3$ . In the PB and PBL methods, the series for the function  $2 + \gamma_\lambda$  was summed using the fourth-order approximant  $[3/1]$ , which is the best (summable) approximant for these methods, while in the CPB method, all the approximants other than  $[N/0]$  turned out to be nonsummable, and only the simple CM method was therefore realized.

The averaging procedure for the results obtained using the PB method with the approximants of order  $N \geq 4$  taken into account allows obtaining the following values of the charge  $g^*$  at the fixed point and of the exponent  $z$ :

$$\begin{aligned} g^* &= 1.7987 \pm 0.0201, & z &= 2.0847 \pm 0.0019 \quad (d = 2); \\ g^* &= 1.4270 \pm 0.0074, & z &= 2.0171 \pm 0.0002 \quad (d = 3), \end{aligned}$$

Using the PBL method gives

$$\begin{aligned} g^* &= 1.7012 \pm 0.0115, & z &= 2.0757 \pm 0.0010 \quad (d = 2); \\ g^* &= 1.4125 \pm 0.0046, & z &= 2.0168 \pm 0.0001 \quad (d = 3) \end{aligned}$$

Using the CPB method gives

$$\begin{aligned} g^* &= 1.8193, & z &= 2.0922 \quad (d = 2); \\ g^* &= 1.4231 \pm 0.0019, & z &= 2.0372 \pm 0.0001 \quad (d = 3). \end{aligned}$$

We note that all the obtained values of the charge  $g^*$  at the fixed point for  $d = 3$  agree well with the result  $g^* = 1.416 \pm 0.005$  obtained by Le Guillou and Zinn-Justin [7].

Averaging the above values obtained by the different methods finally gives

$$\begin{aligned} g^* &= 1.4209 \pm 0.0035, & z &= 2.0237 \pm 0.0055 \quad (d = 3); \\ g^* &= 1.7731 \pm 0.0297, & z &= 2.0842 \pm 0.0039 \quad (d = 2). \end{aligned}$$

In the case where the critical behavior of disordered systems is described, the theory series for the functions  $\beta_g$ ,  $\beta_v$  and  $\gamma_\lambda$  (see (17)) are two-parameter series:

$$F(v, g) = \sum_{i,j} C_{i,j} v^i g^j \quad (i, j = 0, 1, 2, \dots). \quad (27)$$

The methods described above cannot be applied to such series directly. They must be modified for the case of several variables. One that can be thus modified is called the

TABLE III: Results obtained by the PB, PBL, and CPB methods in calculating the fixed points and the dynamical critical exponent  $z$  for the disordered Ising model

		N=2	N=3	N=4		N=5			N=6	
[L/M]		[1/1]	[1/2]	[2/2]	[3/1]	[1/4]	[3/2]	[4/1]	[4/2]	[5/1]
PB	$v^*$	-0.5868	-0.6894	-0.7383	-0.6991	-0.6394	-0.7146	-0.7201	-0.7125	
	$g^*$	2.3900	2.2402	2.2743	2.2411	2.2497	2.2640	2.2705	2.2583	
	$z$	2.1509	2.1756	2.1878	2.1780	2.1633	2.1819	2.1833	2.1814	
PBL	$v^*$	-0.6862	-0.7001	-0.7000	-0.7142	-0.6900	-0.7142	-0.7089	-0.7141	
	$g^*$	2.3895	2.2499	2.2499	2.2459	2.2601	2.2638	2.2646	2.2600	
	$z$	2.1757	2.1787	2.1787	2.1822	2.1762	2.1823	2.1809	2.1822	
CPB	$v^*$		-0.6800	-0.6700	-0.6808	-0.7302	-0.7239		-0.7321	-0.7321
	$g^*$		2.1654	2.1342	2.2562	2.3079	2.2671		2.2763	2.2799
	$z$		2.1716	2.1691	2.1710	2.1844	2.1831		2.1854	2.1853

$\lambda$ -method. The essence of this method consists in introducing a generalized series

$$\tilde{F}(v, g; \lambda) = \sum_{n=0}^{\infty} \tilde{C}_n(v, g) \lambda^n, \quad \tilde{C}_n(v, g) = \sum_{i,j} C_{i,j} v^i g^j \delta_{i+j,n}, \quad (28)$$

which converges to the original series (27) for  $\lambda = 1$ . We can assume that such a series depends on the single variable  $\lambda$  and its coefficients are functions of  $u$  and  $v$ . The Borel transform of series (27) has the form In the  $\lambda$ -method, the Borel image is reduced to a function depending only on the single variable  $\lambda$ , the Padé approximant method is then applied to this function, and the integral is calculated. In the CM method applied to two-parameter series, we use the change  $\delta = v/u$ , which permits rewriting series (27) as

$$F(v, g)|_{g=\delta v} = \sum_{n=0}^{\infty} \tilde{C}_n(\delta) v^n. \quad (29)$$

The Borel transform then becomes

$$F(v, \delta) = \int_0^{\infty} dt e^{-t} B(v t^b, \delta), \quad B(v, \delta) = \sum_{n=0}^{\infty} \frac{\tilde{C}_n(\delta)}{\Gamma(nb+1)} v^n. \quad (30)$$

The conformal map is determined by the formula

$$u(v) = \frac{\sqrt{1 + \alpha(\delta)v} - 1}{\sqrt{1 + \alpha(\delta)v} + 1}. \quad (31)$$

When the CM method is applied to the series of the critical behavior theory for disordered systems, the problem becomes more complicated because the asymptotic behavior of the series is unknown in this case. In [9], for the Ising model, it was proposed to take  $\alpha(\delta)$  in the form

$$\alpha(\delta) = a(9/8 + \delta) \quad (\delta < 0, \delta > 4),$$

where  $a = 0.14777422$  for the three-dimensional Ising model.

We now consider the application of these two methods to series (17), which characterize the nonequilibrium critical behavior of the disordered Ising model. In Table III, we present the results obtained by these summation methods in calculating the coordinates of the fixed point, which determines the critical behavior of disordered systems, and the exponent  $z$ . In the PB and PBL methods, we summed the series for the function  $2 + \gamma_\lambda$  using the third-order approximant  $[2/1]$ , which is the best for these methods, and in the CPB method, we used the approximant  $[1/2]$ . It follows from Table III that all these methods give close values for the vertices at the fixed points and for the exponent  $z$ .

We now apply the averaging procedure to the obtained results using all the values of the approximants  $[L/M]$  except  $[1/1]$ . The averages of the fixed-point coordinates and the exponent  $z$  obtained using the PB method are

$$v^* = -0.7019 \pm 0.0111, \quad g^* = 2.2569 \pm 0.0048, \quad z = 2.1788 \pm 0.0027,$$

Using the PBL method gives

$$v^* = -0.7059 \pm 0.0033, \quad g^* = 2.2563 \pm 0.0026, \quad z = 2.1802 \pm 0.0008$$

Using the CPB method gives

$$v^* = -0.7070 \pm 0.0100, \quad g^* = 2.2410 \pm 0.0227, \quad z = 2.1786 \pm 0.0026.$$

The close values of the fixed-point coordinates and the exponent  $z$  obtained by different methods and practically coinciding within the accuracy confirm the reliability of the obtained results. Averaging these values obtained by different methods finally gives

$$v^* = -0.7049 \pm 0.0013, \quad g^* = 2.2514 \pm 0.0042, \quad z = 2.1792 \pm 0.0004.$$

The dynamical critical exponent  $z$  was calculated in [5] using the Chisholm–Borel method and  $z = 2.1653$  was obtained. This value of the exponent agrees rather well with the results obtained here.



## IV. CONCLUSION

In this paper, we first successively applied the PB, PBL, and CM summation methods with subsequent use of the Padé approximation (the CPB method) to determine the values of the dynamical critical exponent  $z$  for homogeneous two- and three-dimensional systems described by the Ising model and for disordered Ising-like systems. Our analysis was based only on the now available short series of the theory corresponding to the results of the four-loop description of the critical dynamics of homogeneous systems and the three-loop description of disordered systems [4, 5].

We compare the obtained results with the experimental data and the results of Monte Carlo computer simulations of the critical dynamics. In [20], the critical dynamics of the homogeneous Ising antiferromagnet  $FeF_2$  was investigated experimentally, and the value  $z = 2.1 \pm 0.1$  was obtained, which in the framework of a sufficiently wide interval of measurement error for  $z$  does not contradict our results of the field theory calculations using the summation methods. The Monte Carlo numerical investigations of the critical dynamics of the homogeneous three-dimensional Ising model for different data given in the literature gave the following values of the dynamical critical exponent:  $z = 1.99 \pm 0.02$  [21],  $z = 1.97 \pm 0.08$  [22],  $z = 2.04 \pm 0.03$  [23],  $z = 2.04 \pm 0.01$  [24] and  $z = 2.032 \pm 0.004$  [25]. These values of the exponent  $z$  agree well with the results obtained here and confirm that the our predicted value  $z = 2.0237 \pm 0.0055$  is sufficiently reliable.

As for the two-dimensional homogeneous Ising model, the numerical investigations of its critical dynamics, in contrast to similar investigations of the three-dimensional model, are characterized by a much wider range of the values obtained for the dynamical critical exponent:  $z = 2.14 \pm 0.02$  [26],  $z = 2.13 \pm 0.03$  [27],  $z = 2.076 \pm 0.005$  [28],  $z = 2.24 \pm 0.04$  [29],  $z = 2.24 \pm 0.07$  [30],  $z = 2.16 \pm 0.04$  [31] and  $z = 2.1667 \pm 0.0005$  [32]. This range of  $z$  is wide mainly because the critical behavior of the two-dimensional Ising model is characterized by both much greater amplitudes of the order parameter fluctuations in the critical domain (compared with the three-dimensional model) and more significant effects of the critical delay. The high-temperature expansion method used to describe the critical dynamics of the two-dimensional Ising model gives the value  $z = 2.125$  [33]. The values of the exponent  $z$  obtained for the two-dimensional Ising model in the cited papers are in the rather wide interval  $2.07 \leq z \leq 2.31$ , while the values obtained here are on the lower boundary of this

interval. This makes the results obtained here more valuable as reference points in future studies.

We now compare the values of the dynamical critical exponent for disordered systems calculated in this paper with the results of computer simulations of the critical dynamics of the three-dimensional disordered Ising model:  $z = 2.19 \pm 0.07$  for systems with spin concentration  $p = 0.95$ ,  $z = 2.20 \pm 0.08$  for  $p = 0.8$ ,  $z = 2.58 \pm 0.09$  for  $p = 0.6$  and  $z = 2.65 \pm 0.12$  for  $p = 0.4$  [22];  $z = 2.16 \pm 0.01$  for  $p = 0.95$ ,  $z = 2.232 \pm 0.004$  for  $p = 0.9$ ,  $z = 2.38 \pm 0.01$  for  $p = 0.8$  and  $z = 2.93 \pm 0.03$  for  $p = 0.6$  [34]. These results of critical dynamics simulations agree rather well with the results of field theory calculations using the summation methods only for weakly disordered systems with  $p \geq 0.8$ , while a noticeable difference between the results is observed for strongly disordered systems. We note that the results of the RG description of the critical behavior of disordered systems in this case hold only in the domain of weak disorder. To explain the dependence of the exponent  $z$  on the value of the structural disorder observed in computer simulations, the hypothesis of graduated universality was proposed in [22], according to which several types of different critical behavior can be observed in systems with spin concentrations greater than the spin percolation threshold depending on whether there exists only one spin percolation cluster in the system as in the case of weakly disordered systems or there is an admixture percolation cluster in addition to the spin percolation cluster as in the case of strongly disordered systems with transient regimes between the domains.

Starting from the concept that the critical behavior of disordered systems is universal and that the asymptotic value (as  $L \rightarrow \infty$ ) of the exponent  $z$  is independent of the degree of disorder, the author of [34] obtained the asymptotic value  $z = 2.4 \pm 0.1$  using the effective values of the exponent listed above. But this value of the exponent is strongly inconsistent with the results obtained here. A numerical study of the critical dynamics of the three-dimensional Ising model with the spin concentration varying in a wide interval was analyzed in [35]. Assuming that the critical behavior of disordered systems is universal, the authors of [35] obtained the asymptotic value of the exponent  $z = 2.62 \pm 0.07$  taking the effects of influence of the leading corrections to the scaling dependence for the dynamical susceptibility of the system into account. In this case, the value of the exponent of the scaling correction,  $\omega = 0.50 \pm 0.13$ , obtained in [35] is strongly inconsistent with the results of the field theory calculations of the static critical exponents, which were done using the summation methods

in [9] and gave  $\omega = 0.25 \pm 0.10$ . This value is also inconsistent with the results of numerical investigations of the static critical behavior of the same model [36] also done taking the effects of influence of the leading corrections to the scaling dependence of thermodynamic quantities and the correlation functions with  $\omega = 0.37 \pm 0.06$  into account. In the approximations realized in [35], the results for weakly disordered systems were characterized by the largest errors. We also note that the value of the dynamical critical exponent  $z$  obtained in [35] is even more strongly inconsistent with our results here.

The critical dynamics of the disordered Ising antiferromagnet  $Fe_{0.46}Zn_{0.54}F_2$  was studied experimentally in [20], where  $z = 1.7 \pm 0.2$  was obtained. This result conflicts with both the results of the field theory calculations using the summation methods and the above results of computer simulations. This inconsistency can possibly be explained by a high concentration of nonmagnetic admixture atoms in the sample under study and by the presence of large-scale inhomogeneities in this sample, which significantly affects the characteristics of the nonequilibrium critical behavior. But in the other experimental paper concerning the critical dynamics of weakly diluted Ising magnet  $Fe_{0.9}Zn_{0.1}F_2$  [37], the value of the dynamical critical exponent  $z = 2.18 \pm 0.10$  was obtained as the result of high-precession measurement of the dynamical widening of Mössbauer lines using the Mössbauer spectroscopy method. This value agrees very well with the results of our calculations. Nevertheless, there is a strong need for additional investigations of the critical dynamics of disordered Ising-like systems, both experimentally and numerically. The results of calculations of the dynamical critical exponent for disordered systems obtained here will be reference points in such investigations.

### Acknowledgments

This work was supported by the Russian Foundation for Basic Research through Grants (No. 04-02-17524, No. 04-02-39000 and No. 05-02-16188), and by Grant No. MK-8738.2006.2 of Russian Federation President.

- 
- [1] P.C. Hohenberg and B.I. Halperin, *Rev. Modern Phys.*, **49**, 435 (1977).
  - [2] R. Bausch, H.K. Janssen, and H. Wagner, *Z. Phys. B*, **24**, 113 (1976);  
C. De Dominicis, E. Brezin, and J. Zinn-Justin, *Phys. Rev. B*, **12**, 4945 (1975);

- N.V. Antonov and A.N. Vasil'ev, *Theor. Math. Phys.*, **60**, 671 (1984);
- B.I. Halperin, P.C. Hohenberg, and S. Ma, *Phys. Rev. Lett.*, **29**, 1548 (1972);
- B.I. Halperin, P.C. Hohenberg, E.D. Siggia, and S. Ma, *Phys. Rev. B*, **13**, 2110 (1976).
- [3] V.V. Prudnikov and A.N. Vakilov, *JETP*, **74**, 990 (1992).
- [4] V.V. Prudnikov, A.V. Ivanov, and A.A. Fedorenko, *JETP Letters*, **66**, 835 (1997).
- [5] V. V. Prudnikov et al., *JETP*, **87**, 527 (1998).
- [6] G.A. Baker, B.G. Nickel, M.S. Green, and D.I. Meiron, *Phys. Rev. Lett.*, **36**, 1351 (1976);  
*Phys. Rev. B*, **17**, 1365 (1978).
- [7] J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. Lett.*, **39**, 95 (1977); *Phys. Rev. B*, **21**, 3976 (1980).
- [8] S.A. Antonenko and A.I. Sokolov, *Phys. Rev. B*, **51**, 1894 (1995).
- [9] A. Pelissetto and E. Vicari, *Phys. Rev. B*, **62**, 6393 (2000).
- [10] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon, Oxford, 1008 (1996); A. N. Vasil'ev, *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* [in Russian], PNPI, St. Petersburg (1998); English transl., CRC Press, Boca Raton, Fl. (2004).
- [11] V.S. Dotsenko, *Phys. Usp.*, **138**, 457 (1995).
- [12] L.N. Lipatov, *JETP*, **45**, 216 (1977).
- [13] E. Brézin, J.C. Le Guillou, and J. Zinn-Justin, *Phys. Rev. D*, **15**, 1544 (1977).
- [14] G.H. Hardy, *Divergent Series*, Clarendon, Oxford (1949);
- [15] D.I. Kazakov, O.V. Tarasov, and D.V. Shirkov, *Theor. Math. Phys.*, **38**, 9 (1979);  
D.I. Kazakov and V.S. Popov, *JETP*, **95**, 581 (2002).
- [16] I.M. Suslov, *JETP*, **93**, 1 (2001).
- [17] J. Honkonen, M.V. Komarova, and M.Yu. Nalimov, *Nucl. Phys. B*, **714**, 292 (2005);  
*Nucl. Phys. B*, **707**, 493 (2005); *hep-th/0406168* (2004).
- [18] C.M. Bender and T.T. Wu, *Phys. Rev.*, **184**, 1231 (1969).
- [19] J. Cizek and E.R. Vrskey, *Int. J. Quant. Chem.*, **21**, 27 (1982).
- [20] D.P. Belanger, et al., *Journ. de Phys.*, **49**, C8-1229 (1988).
- [21] R.B. Pearson, J.L. Richardson, and D. Toussaint, *Phys. Rev. B*, **31**, 4472 (1985).
- [22] V.V. Prudnikov and A.N. Vakilov, *JETP*, **76**, 469 (1993).
- [23] S. Wansleben and D.P. Landau, *Phys. Rev. B*, **43**, 6006 (1991).

- [24] U. Gropengiessen, *Phys. A*, **213**, 308 (1995).
- [25] P. Grassberger, *Phys. A*, **214**, 547 (1995).
- [26] C. Kalle, *J. Phys. A*, **17**, L801 (1984).
- [27] J.K. Williams, *J. Phys. A*, **18**, 49 (1985).
- [28] M. Mori and Y. Tsuda, *Phys. Rev. B*, **37**, 5444 (1988).
- [29] P.H. Poole and N. Jan, *J. Phys. A*, **23**, L453 (1990).
- [30] V.V. Prudnikov and O.N. Markov, *Europhys. Lett.*, **29**, 245 (1995).
- [31] F. Wang, N. Hatane, and M. Suzuki, *J. Phys. A*, **28**, 4543 (1995).
- [32] M.P. Nightingale and H.W.J. Blöte, *Phys. Rev. B*, **62**, 1089 (2000).
- [33] Z. Racz and M.F. Collins, *Phys. Rev. B*, **13**, 3074 (1976).
- [34] H.-O. Heuer, *J. Phys. A*, **26**, L341 (1993).
- [35] G. Parisi, F. Ricci-Tersenghi, and J.J. Ruiz-Lorenzo, *Phys. Rev. E*, **60**, 5198 (1999).
- [36] H.G. Ballesteros et al., *Phys. Rev. B*, **58**, 2740 (1998).
- [37] N. Rosov, C. Hohenemser, and M. Eibschutz, *Phys. Rev. B*, **46**, 3452 (1992).